Prove

[07-04-23-AT-11]

2. $\csc x \sin x = \sec x \cos x$

3. $\cot x \sin x = \cos x$

4. $\tan x = -\tan(-x)$

5. $\dfrac{\sin^2 x}{\cos x} = \sec x - \cos x$

6. $\dfrac{\csc x}{\cos x} = \tan x + \cot x$

7. $(\cos^2 x)(\cot^2 x + 1) = \cot^2 x$

8. $\dfrac{\sin \alpha \csc \alpha}{\cot \alpha} = \tan \alpha$

9. $\dfrac{\sin^2 u - \cos^2 u}{\sin u \cos u} = \tan u - \cot u$

10. $\dfrac{\sec \theta - \csc \theta}{\sec \theta \csc \theta} = \sin \theta - \cos \theta$

11. Using $\cos(x + y) = \cos x \cos y - \sin x \sin y$, show that $\cos(x + 2\pi) = \cos x$.

12. Using $\sin(x + y) = \sin x \cos y + \cos x \sin y$, show that $\sin(x + \pi) = -\sin x$.

In Problems 13 and 14, verify each identity for the indicated value.

13. $\cos 2x = 1 - 2\sin^2 x, x = 30°$

14. $\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}, x = \dfrac{\pi}{2}$

☆ **15.** Write $\sin 8t \sin 5t$ as a sum or difference.

☆ **16.** Write $\sin w + \sin 5w$ as a product.

Verify each identity in Problems 17–20.

17. $\dfrac{1 - \cos^2 t}{\sin^3 t} = \csc t$

18. $\dfrac{(\cos \alpha - 1)^2}{\sin^2 \alpha} = \dfrac{1 - \cos \alpha}{1 + \cos \alpha}$

19. $\dfrac{1 - \tan^2 x}{1 - \tan^4 x} = \cos^2 x$

27. $\dfrac{\sin x}{1 - \cos x} = (\csc x)(1 + \cos x)$

28. $\dfrac{1 - \tan^2 x}{1 - \cot^2 x} = 1 - \sec^2 x$

29. $\tan(x + \pi) = \tan x$

30. $1 - (\cos \beta - \sin \beta)^2 = \sin 2\beta$

31. $\dfrac{\sin 2x}{\cot x} = 1 - \cos 2x$

32. $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

33. $2 \csc 2x = \tan x + \cot x$

34. $\csc x = \dfrac{\cot(x/2)}{1 + \cos x}$

35. $\dfrac{\sin(x - y)}{\sin(x + y)} = \dfrac{\tan x - \tan y}{\tan x + \tan y}$

36. $\csc 2x = \dfrac{\tan x + \cot x}{2}$

37. $\dfrac{2 - \sec^2 x}{\sec^2 x} = \cos 2x$

38. $\tan \dfrac{x}{2} = \dfrac{\sec x - 1}{\tan x}$

☆ **39.** $\dfrac{\sin t + \sin 5t}{\cos t + \cos 5t} = \tan 3t$

☆ **40.** $\dfrac{\sin x + \sin y}{\cos x - \cos y} = -\cot \dfrac{x - y}{2}$

☆ **41.** $\dfrac{\cos x - \cos y}{\cos x + \cos y} = -\tan \dfrac{x + y}{2} \tan \dfrac{x - y}{2}$

Answers

2. $\csc x \sin x = \frac{1}{\sin x} \sin x$ — Reciprocal identity

$= 1$ — Algebra

$= \frac{1}{\cos x} \cos x$ — Algebra

$= \sec x \cos x$ — Reciprocal identity

3. $\cot x \sin x = \frac{\cos x}{\sin x} \sin x$ — Quotient identity

$= \cos x$ — Algebra

4. $\tan x = \frac{\sin x}{\cos x}$ — Quotient identity

$= \frac{-\sin(-x)}{\cos(-x)}$ — Identities for negatives

$= -\tan(-x)$ — Quotient identity

5. $\frac{\sin^2 x}{\cos x} = \frac{1 - \cos^2 x}{\cos x}$ — Pythagorean identity

$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$ — Algebra

$= \frac{1}{\cos x} - \cos x$ — Algebra

$= \sec x - \cos x$ — Reciprocal identity

6. $\dfrac{\csc x}{\cos x} = \dfrac{\frac{1}{\sin x}}{\cos x}$ Reciprocal identity

$= \dfrac{1}{\sin x} \div \cos x$ Algebra

$= \dfrac{1}{\sin x} \cdot \dfrac{1}{\cos x}$ Algebra

$= \dfrac{1}{\sin x \cos x}$ Algebra

$= \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ Pythagorean identity

$= \dfrac{\sin^2 x}{\sin x \cos x} + \dfrac{\cos^2 x}{\sin x \cos x}$ Algebra

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$ Algebra

$= \tan x + \cot x$ Quotient identities

7. $\cos^2 x (1 + \cot^2 x) = \cos^2 x \csc^2 x$ Pythagorean identity

$= \cos^2 x \dfrac{1}{\sin^2 x}$ Reciprocal identity

$= \dfrac{\cos^2 x}{\sin^2 x}$ Algebra

$= \cot^2 x$ Quotient identity

8. $\dfrac{\sin \alpha \csc \alpha}{\cot \alpha} = \dfrac{\sin \alpha \cdot \frac{1}{\sin \alpha}}{\cot \alpha}$ Reciprocal identity

$= \dfrac{1}{\cot \alpha}$ Algebra

$= \tan \alpha$ Reciprocal identity

9. $\dfrac{\sin^2 u - \cos^2 u}{\sin u \cos u} = \dfrac{\sin^2 u}{\sin u \cos u} - \dfrac{\cos^2 u}{\sin u \cos u}$ Algebra

$= \dfrac{\sin u}{\cos u} - \dfrac{\cos u}{\sin u}$ Algebra

$= \tan u - \cot u$ Quotient identities

10. $\dfrac{\sec \theta - \csc \theta}{\sec \theta \csc \theta} = \dfrac{\sec \theta}{\sec \theta \csc \theta} - \dfrac{\csc \theta}{\sec \theta \csc \theta}$ Algebra

$= \dfrac{1}{\csc \theta} - \dfrac{1}{\sec \theta}$ Algebra

$= \sin \theta - \cos \theta$ Reciprocal identities

11. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x + 2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi$

$= \cos x(1) - \sin x(0)$

$= \cos x$

12. $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x + y) = \sin x \cos \pi + \cos x \sin \pi$

$= \sin x(-1) + \cos x(0)$

$= -\sin x$

13. $\cos 2x = \cos 2(30°) = \cos 60° = \frac{1}{2}$

$1 - 2\sin^2 x = 1 - 2\sin^2(30°)$

$= 1 - 2(\sin 30°)^2$

$= 1 - 2\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2}$

$= \frac{1}{2}$

14. $\sin\frac{x}{2} = \sin\frac{\pi/2}{2} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Since $\frac{\pi}{4}$ is in the first quadrant, the sign of the square root is chosen to be positive.

$$\sqrt{\frac{1 - \cos x}{2}} = \sqrt{\frac{1 - \cos \pi/2}{2}} = \sqrt{\frac{1 - 0}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

15. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$ Let $x = 8t$ and $y = 5t$

$\sin 8t \sin 5t = \frac{1}{2}[\cos(8t - 5t) - \cos(8t + 5t)] = \frac{1}{2}(\cos 3t - \cos 13t) = \frac{1}{2}\cos 3t - \frac{1}{2}\cos 13t$

16. $\sin x + \sin y = 2\sin\frac{x + y}{2}\cos\frac{x - y}{2}$ Let $x = w$ and $y = 5w$

$\sin w + \sin 5w = 2\sin\frac{w + 5w}{2}\cos\frac{w - 5w}{2} = 2\sin 3w \cos(-2w) = 2\sin 3w \cos 2w$

17. $\frac{1 - \cos^2 t}{\sin^3 t} = \frac{\sin^2 t}{\sin^3 t}$ Pythagorean identity

$= \frac{1}{\sin t}$ Algebra

$= \csc t$ Reciprocal identity

18. $\dfrac{(\cos\alpha - 1)^2}{\sin^2\alpha} = \dfrac{(\cos\alpha - 1)^2}{1 - \cos^2\alpha}$ Pythagorean identity

$= \dfrac{(\cos\alpha - 1)(\cos\alpha - 1)}{(1 - \cos\alpha)(1 + \cos\alpha)}$ Algebra

$= \dfrac{(-1)(\cos\alpha - 1)}{1 + \cos\alpha}$ Algebra

$= \dfrac{1 - \cos\alpha}{1 + \cos\alpha}$ Algebra

Key algebraic steps: $\dfrac{(b-1)^2}{1-b^2} = \dfrac{(b-1)(b-1)}{(1-b)(1+b)} = \dfrac{(-1)(b-1)}{1+b} = \dfrac{1-b}{1+b}$

19. $\dfrac{1 - \tan^2 x}{1 - \tan^4 x} = \dfrac{1 - \tan^2 x}{(1)^2 - (\tan^2 x)^2}$ Algebra

$= \dfrac{1 - \tan^2 x}{(1 - \tan^2 x)(1 + \tan^2 x)}$ Algebra

$= \dfrac{1}{1 + \tan^2 x}$ Algebra

$= \dfrac{1}{\sec^2 x}$ Pythagorean identity

$= \left(\dfrac{1}{\sec x}\right)^2$ Algebra

$= \cos^2 x$ Reciprocal identity

Key algebraic steps: $\dfrac{1-c^2}{1-c^4} = \dfrac{1-c^2}{(1)^2-(c^2)^2} = \dfrac{1-c^2}{(1-c^2)(1+c^2)} = \dfrac{1}{1+c^2}$

20. $\cot^2 x \cos^2 x = (\csc^2 x - 1)\cos^2 x$ Pythagorean identity

$= \csc^2 x \cos^2 x - \cos^2 x$ Algebra

$= \left(\dfrac{1}{\sin x}\right)^2 \cos^2 x - \cos^2 x$ Reciprocal identity

$= \left(\dfrac{\cos x}{\sin x}\right)^2 - \cos^2 x$ Algebra

$= \cot^2 x - \cos^2 x$ Quotient identity

27. $\dfrac{\sin x}{1-\cos x} = \dfrac{\sin x}{1-\cos x}\dfrac{1+\cos x}{1+\cos x}$ Algebra

$= \dfrac{\sin x(1+\cos x)}{1-\cos^2 x}$ Algebra

$= \dfrac{\sin x(1+\cos x)}{\sin^2 x}$ Pythagorean identity

$= \dfrac{\sin x}{\sin^2 x}(1+\cos x)$ Algebra

$= \dfrac{1}{\sin x}(1+\cos x)$ Algebra

$= \csc x(1+\cos x)$ Reciprocal identity

28. $\dfrac{1-\tan^2 x}{1-\cot^2 x} = \dfrac{1-\tan^2 x}{1-\dfrac{1}{\tan^2 x}}$ Reciprocal identity

$= \dfrac{\tan^2 x\,(1-\tan^2 x)}{\tan^2 x\left(1-\dfrac{1}{\tan^2 x}\right)}$ Algebra

$= \dfrac{\tan^2 x\,(1-\tan^2 x)}{\tan^2 x - 1}$ Algebra

$= \dfrac{-\tan^2 x\,(\tan^2 x - 1)}{\tan^2 x - 1}$ Algebra

$= -\tan^2 x$ Algebra

$= -(\sec^2 x - 1)$ Pythagorean identity

$= 1 - \sec^2 x$ Algebra

Key algebraic steps: $\dfrac{1-a^2}{1-\dfrac{1}{a^2}} = \dfrac{a^2(1-a^2)}{a^2\left(1-\dfrac{1}{a^2}\right)} = \dfrac{a^2(1-a^2)}{a^2-1} = \dfrac{-a^2(a^2-1)}{a^2-1} = -a^2$

29. $\tan(x+\pi) = \dfrac{\tan x + \tan \pi}{1 - \tan x \tan \pi}$ Sum identity

$= \dfrac{\tan x + 0}{1 - \tan x \cdot 0}$ Known values

$= \tan x$ Algebra

30. $1-(\cos\beta - \sin\beta)^2 = 1 - (\cos^2\beta - 2\sin\beta\cos\beta + \sin^2\beta)$ Algebra

$= 1 - \cos^2\beta + \sin^2\beta - 2\sin\beta\cos\beta)$ Algebra

$= 1 - (1 - 2\sin\beta\cos\beta)$ Pythagorean identity

$= 1 - 1 + 2\sin\beta\cos\beta$ Algebra

$= 2\sin\beta\cos\beta$ Algebra

$= \sin 2\beta$ Double-angle identity

31. $\dfrac{\sin 2x}{\cot x} = \dfrac{2\sin x\cos x}{\cot x}$ Double-angle identity

$= \dfrac{2\sin x\cos x}{\dfrac{\cos x}{\sin x}}$ Quotient identity

$= 2\sin x\cos x + \dfrac{\cos x}{\sin x}$ Algebra

$= 2\sin x\cos x \cdot \dfrac{\sin x}{\cos x}$ Algebra

$= 2\sin^2 x$ Algebra

$= 2\sin^2 x - 1 + 1$ Algebra

$= 1 + (2\sin^2 x - 1)$ Algebra

$= 1 - (1 - 2\sin^2 x)$ Algebra

$= 1 - \cos 2x$ Double-angle identity

32. $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$ Quotient identity

$= \dfrac{\cos^2 x \cdot 2\dfrac{\sin x}{\cos x}}{\cos^2 x \cdot 1 + \cos^2 x \cdot \dfrac{\sin^2 x}{\cos^2 x}}$ Algebra

$= \dfrac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$ Algebra

$= \dfrac{2\sin x \cos x}{1}$ Pythagorean identity

$= 2\sin x \cos x$ Algebra

$= \sin 2x$ Double-angle identity

33. $2\csc 2x = \dfrac{2}{\sin 2x}$ Reciprocal identity

$= \dfrac{2}{2\sin x \cos x}$ Double-angle identity

$= \dfrac{1}{\sin x \cos x}$ Algebra

$= \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ Pythagorean identity

$= \dfrac{\sin^2 x}{\sin x \cos x} + \dfrac{\cos^2 x}{\sin x \cos x}$ Algebra

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$ Algebra

$= \tan x + \cot x$ Quotient identities

34. $\dfrac{\cot\dfrac{x}{2}}{1+\cos x} = \cot\dfrac{x}{2} \cdot \dfrac{1}{1+\cos x}$ Algebra

$= \dfrac{1}{\tan\dfrac{x}{2}} \cdot \dfrac{1}{1+\cos x}$ Reciprocal identity

$= \dfrac{1}{\dfrac{\sin x}{1+\cos x}} \cdot \dfrac{1}{1+\cos x}$ Half-angle identity

$= \dfrac{1+\cos x}{\sin x} \cdot \dfrac{1}{1+\cos x}$ Algebra

$= \dfrac{1}{\sin x}$ Algebra

$= \csc x$ Reciprocal identity

35. $\dfrac{\sin(x-y)}{\sin(x+y)} = \dfrac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y}$ — Sum and difference identities

$= \dfrac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}$ — Algebra

$= \dfrac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}$ — Algebra

$= \dfrac{\tan x - \tan y}{\tan x + \tan y}$ — Quotient identity

36. $\csc 2x = \dfrac{1}{\sin 2x}$ — Reciprocal identity

$= \dfrac{1}{2 \sin x \cos x}$ — Double-angle identity

$= \dfrac{\sin^2 x + \cos^2 x}{2 \sin x \cos x}$ — Pythagorean identity

$= \dfrac{\sin^2 x}{2 \sin x \cos x} + \dfrac{\cos^2 x}{2 \sin x \cos x}$ — Algebra

$= \dfrac{\sin x}{2 \cos x} + \dfrac{\cos x}{2 \sin x}$ — Algebra

$= \dfrac{1}{2}\dfrac{\sin x}{\cos x} + \dfrac{1}{2}\dfrac{\cos x}{\sin x}$ — Algebra

$= \dfrac{1}{2}\tan x + \dfrac{1}{2}\cot x$ — Quotient identities

$= \dfrac{\tan x + \cot x}{2}$ — Algebra

37. $\dfrac{2 - \sec^2 x}{\sec^2 x} = \dfrac{1 - \dfrac{1}{\cos^2 x}}{\dfrac{1}{\cos^2 x}}$ — Reciprocal identity

$= \dfrac{\cos^2 x \cdot 2 - \cos^2 x \cdot \dfrac{1}{\cos^2 x}}{\cos^2 x \cdot \dfrac{1}{\cos^2 x}}$ — Algebra

$= \dfrac{2 \cos^2 x - 1}{1}$ — Algebra

$= 2 \cos^2 x - 1$ — Algebra

$= \cos 2x$ — Double-angle identity

38. $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$ — Half-angle identity

$= \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{\frac{\sin x}{\cos x}}$ — Algebra

$= \frac{\sec x - \frac{\cos x}{\cos x}}{\frac{\sin x}{\cos x}}$ — Reciprocal identity

$= \frac{\sec x - 1}{\frac{\sin x}{\cos x}}$ — Algebra

$= \frac{\sec x - 1}{\tan x}$ — Quotient identity

39. $\frac{\sin t + \sin 5t}{\cos t + \cos 5t} = \frac{2\sin\frac{t+5t}{2}\cos\frac{t-5t}{2}}{2\cos\frac{t+5t}{2}\cos\frac{t-5t}{2}}$ — Sum-product identities

$= \frac{2\sin 3t\cos(-2t)}{2\cos 3t\cos(-2t)}$ — Algebra

$= \frac{\sin 3t}{\cos 3t}$ — Algebra

$= \tan 3t$ — Quotient identity

40. $\frac{\sin x + \sin y}{\cos x - \cos y} = \frac{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}}{-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}$ — Sum-product identities

$= -\frac{\cos\frac{x-y}{2}}{\sin\frac{x-y}{2}}$ — Algebra

$= -\cot\frac{x-y}{2}$ — Quotient identity

41. $\frac{\cos x - \cos y}{\cos x + \cos y} = \frac{-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}}$ — Sum-product identities

$= -\frac{\sin\frac{x+y}{2}}{\cos\frac{x+y}{2}}\frac{\sin\frac{x-y}{2}}{\cos\frac{x-y}{2}}$ — Algebra

$= -\tan\frac{x+y}{2}\tan\frac{x-y}{2}$ — Quotient identity